STATISTICAL APPROACH TO THE MECHANICAL BEHAVIOR OF GRANULAR MEDIA

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Abstract We discuss the quasistatic rheology of ideal granular media consisting of rigid discs interacting via Coulomb law of friction and perfectly inelastic collisions. The macroscopic description of the rheology of quasistatic deformation of such media is rigid-plastic with hardening laws parametrized with internal variables which have to characterize the geometry of the assembly. A phenomenological approach is proposed along these lines. An outline of a microscopic/macroscopic derivation of the required characteristics is presented. Finally, we list some possible effects of fluctuations which may limit the precise quantitative success of this approach.

1. INTRODUCTION

The quasi-static behavior of granular materials is already a mature field in which a number of elasto-plastic models reproduce very accurately the available experimental tests. They allow to design civil engineering structures with confidence. However, this description is essentially based on extensions of the elasto-plasticity of other materials rather than a microscopic modelling of a granular assembly.(Mróz 98)

On the other hand, focussing on the details of particle interactions, a considerable progress has been made in recent years through various algorithms which allow to describe different regimes of granular flows. (see e.g. (Bardet 98; Kishino 99) for a recent review.) The discrete modelling of granular media allows nowadays very accurate simulation of very stiff particles up to 10^4 particles for a significant cumulative strain. In cases where the accurate modelling of the contact and of the particle properties is less stringent, more than one million grains can be taken into account. The interest of this approach is that it allows also to follow the response of a granular assembly subjected to uniform stress or strain. In this context, the use of bi or tri-periodic (resp. in two or three dimensions) boundary conditions is an important achievement allowing to reduce significantly the role of walls. We are thus now in a suitable position to answer questions pertaining to the characterization of the geometry of the packing (and hence eventually to answer unsolved issues such as the orientation of localization bands), to address the role of fluctuations, to measure the effective stress carried by a specific granulometric class in a packing (relevant for fragmentation), ..., issues which are clearly out of reach within today's continuum modelling.

With this motivation in mind, one is naturally invited to have a fresh look at the continuum modelling and to progress in the direction of introducing geometric information in the macroscopic modelling, even if such an approach will inevitably lead first to a deterioration in the accuracy of the macroscopic modelling. The hope is that, after some time and effort, one may achieve a more satisfactory description in terms of connections to the microscopic reality and still with a fair account for experimental tests.

2. A MODEL SYSTEM

In the following, we will focus on the simple model of a granular assembly of rigid discs in two dimensions. These particles interact only via a hard core potential (no adhesion), and Coulomb friction law with a single coefficient of friction. When numerical simulations are used, a slight polydersity is introduced in order to avoid the crystalization of the system which is a specific feature of two-dimensional systems. However, in our theoretical description we ignore the polydispersity and the particles will be characterized by a single average radius R.

2.1. GENERAL REMARKS CONCERNING THE MACROSCOPIC MODELLING

The particles being considered as rigid and with no adhesion, the macroscopic behavior has to be *rigid-plastic*. Therefore, the domain of admissible stresses $\mathcal{D}$, where the medium is rigid, with its boundary $\partial\mathcal{D}$, where plastic flow may take place, has to be specified. The absence of a stress scale imposes moreover that $\mathcal{D}$ is a *cone* in stress space.

The Coulomb friction law itself can be seen as a rigid-plastic behavior. Within this framework, the Coulomb friction does not obey the normality rule, and hence the macroscopic modelling has to be a *non-associated* plastic behavior. Therefore, the direction of the plastic strain rate has to be specified independently from $\mathcal{D}$.

The plastic flow rule as well as the yield stress have to be defined as a function of the internal variables which characterize the state of the medium. When considered at the microscopic scale, it is obvious that this state is only defined through its *geometry*, i.e. the spatial organization of particles with respect to each other, and hence it is natural to require that internal variables have a simple geometrical interpretation. Although this statement seems quite obvious, it readily disqualifies a number of macroscopic descriptions used at present.

Stated simply, the fundamental difference between a microscopic modelling as compared to a macroscopic one lies in the number of internal variables. Going from the micro to the macro-scale, we wish to preserve only a few of them, enough to characterize precisely the geometrical state of the assembly and not too many so as to be able to have an efficient formulation and a limited number of parameters to adjust through fitting rheological responses or microscopic considerations. As usual, the choice of pertinent state variables is the crucial issue which often results from a compromise between accuracy and simplicity. In order to highlight this compromise, we proceed in three steps. First, we apply the above stated constraints to the formulation of a (trivial) rigid plastic description with no internal variable. Then, we go one step further and incorporate a single scalar internal variable. Finally, we add the fabric to the description. We will see that the incorporation of more and more variable leads to a progressively richer description. We will sketch the outline of a systematic procedure.

Coming back to the macroscopic description, in order to arrive at a complete formulation we have to specify three points: 1) A yield stress parametrized by the internal variables; 2) A plastic flow rule parametrized by the internal variables; 3) A "hardening" law which describes the incremental evolution of internal variables with plastic strain.

3. NO INTERNAL VARIABLE

The most elementary choice is to assume that no internal variable is necessary. In this case, there is obviously no need for a "hardening law". The medium has to be assumed isotropic (otherwise the anisotropy parameters would constitute internal variables). In two dimensions, the requirement of objectivity implies that the domain $\mathcal{D}$ is a function of stress invariants. The easiest case refers to the principal stresses, i.e. a two dimensional description of the stress space. Then, $\mathcal{D}$ is a cone whose axis has to lie along the isotropic pressure direction. Thus, a single parameter is required, that quantifies the relative magnitude of the deviatoric stress with respect to the average stress (trace). This single parameter can be rephrased in terms of the classical friction angle ϕ in the Mohr-Coulomb framework.

For similar reasons, the plastic flow rule consists in defining the relative amount of dilation/contraction with respect to the deviatoric part of the plastic strain rate. Since the macroscopic description should hold for an arbitrary large cumulative strain, the only physically admissible choice is to require that the plastic strain is isochoric, or

$$tr(\dot{\varepsilon}_p) = 0 \tag{1.1}$$

This corresponds to the fact that the internal state of the granular medium is assumed to be unique, and thus it has a well-defined packing fraction that is independent of its prior deformation history. Moreover, along the plane where the Mohr-Coulomb criterion is reached, the axial strain has to be zero. This defines uniquely the relative orientations of the principal axes of the strain rate and stress.

The above considerations are sufficient to fully determine the macroscopic bahaviour of the medium in two dimensions. In particular, this shows that a single scalar parameter is needed, the Mohr-Coulomb friction angle. It is obvious that the resulting description is very crude. Nevertheless, in cases where the cumulative shear strain is large (e.g. surface flow in avalanches, ...), this level of description may prove quite sufficient. The unique state of the medium corresponds to the "critical state" of soil mechanics reached after a sufficiently large deformation. (Schofield and Wroth 68)

4. ONE INTERNAL VARIABLE

Let us now try to add some information about the geometrical state of the medium. The first and most obvious property pertaining to the geometry of a granular packing is the packing fraction c, defined in two dimensions as the fraction of surface covered by the particles. A number of equivalent variants such as void ratio, density of particle centers, global density and porosity may be used instead.

We can proceed as in the previous case but using c as an internal variable for the yield surface and for the plastic flow direction. The yield surface is still parametrized by a friction angle, but now the latter is a continuous (increasing) function $\phi(c)$ of c. Concerning the direction of the incremental plastic strain, dilation and contraction are now admissible. The ratio of the spherical part to the deviatoric part of the plastic strain-rate tensor $\dot{\varepsilon}_p$ defines a *dilation angle*, $\psi(c)$.

$$\sin(\psi) = \frac{\dot{\varepsilon}_1 + \dot{\varepsilon}_2}{\dot{\varepsilon}_1 - \dot{\varepsilon}_2} \tag{1.2}$$

where $\dot{\varepsilon}_{1,2}$ are the two eigenvalues of $\dot{\varepsilon}_p$. In the absence of internal variables we argued that $\psi = 0$. Following the same argument, one infers that there should exist a specific packing fraction c^*, such that

$$\psi(c^*) = 0 \tag{1.3}$$

This packing fraction characterizes then the critical state.

Now, a hardening law has to be specified. It should describe the way c evolves under an increase of plastic strain. In the present case, this evolution law is dictated by the very definition of our internal variable, namely

$$\frac{\dot{c}}{c} = -tr(\dot{\varepsilon}_p) \tag{1.4}$$

This description provides already a more detailed description of the transient stages leading to the critical state. One also recovers the previous model if only large strains are considered. The price to pay for this more accurate description is that we have to specify two functions $\phi(c)$ and $\psi(c)$. There we have different routes at our disposal: either we consider a purely phenomenological approach and thus we try to identify these functions from simple tests(Schofield and Wroth 68) or we may try to relate those functions to a microscopic description of the packing. Eventually, we could also combine both approaches by using part of the information from the micro-scale and identify only a few parameters from experiments. Along the latter direction, one could for instance use Taylor's hypothesis(Taylor 48) in order to relate the two functions $\phi(c)$ and $\psi(c)$ together, and measure then only one of them experimentally. We will come back to this issue after the following section devoted to a richer description of the geometrical state of the medium.

In order to simplify the analysis and the number of parameters to be introduced, we may focus on the neighborhood of the critical state and

Taylor-expand the functions of interest. This leads to a simple three-parameter description, based on $\phi(c^*)$, $d\phi/dc(c = c^*)$, $d\psi/dc(c = c^*)$. (Roux and Radjaï)

An interesting point to note is that this approach naturally leads to the occurence of localization in a dense granular medium. We will however not enter this issue which would require lengthy developments.

5. FABRIC AS STATE VARIABLES

The previous modelling predicts that once the system has reached its critical state, it remains in this state for all direction of shearing. This is not what is observed experimentally. In particular, if the shear is simply reversed, one typically observes a long transient deformation where the system evolves towards a new critical state.

This clearly indicates that a scalar internal variable such as c (which is unable to encode a specific anisotropy of the medium) is insufficient for the description of the state of the medium. From symmetry arguments, the most elementary object which may characterize such an information is a second order tensor. The latter has to be built from a statistical information describing pairs of particles. Moreover, assuming that the required information is local, one should focus on particles in contact.

These remarks point to the internal variables pertaining to the distribution of contact normals. Let us consider the probability distribution of contact normals $p(\theta)$. This is the probability that a given particle has a contact along the direction parametrized by θ, the polar angle of the contact normal $\vec{n}$. The function $p(\theta)$ is π-periodic and it can be Fourier expanded as(Rothenburg and Bathurst 89)

$$p(\theta) = A + B\cos(2(\theta - \theta_p)) + h.o.t. \tag{1.5}$$

Truncation of the Fourier expansion after the second term provides the most salient features of the texture of the medium. In a totally equivalent fashion, one could simply construct the classical fabric tensor $F \equiv \langle \vec{n} \otimes \vec{n} \rangle$ where the brackets denote averaging over all particles in a representative element of volume, and $\otimes$ is the dyadic (tensor) product.(Satake 82) While the fabric tensor is often normalized with respect to the total number of contacts, here we choose to normalize it by the number of particles. In other words, in our case $A = z/\pi$, where z is the coordination number. Assuming that the latter is simply related to the packing fraction, we see that the three descriptions considered above can be seen as retaining more and more terms (0, 1 or 2) in the Fourier expansion of p.

Let us now characterize all the required information to obtain a complete mechanical description of the behaviour using A, B and θ_p as internal variables:

- *Yield stress*: Since the medium has some anisotropy, a single friction angle is no more appropriate. However, in order to keep close to the concepts used within the previous cases, we consider all potential slip planes characterized by a polar angle θ of the normal to the slip plane. The largest ratio of the tangent to the normal stress which can be supported by this plane naturally defines a friction angle that is a function of θ and the above three internal variables. Galilean invariance implies that only the difference $\theta - \theta_p$ is meaningful. We thus have to specify the function

$$\phi = \phi(\theta - \theta_p, A, B) \tag{1.6}$$

- *Plastic strain rate*: Similarly, along each potential slip plane we can characterize the orientation of the relative velocity of two points aligned perpendicular to the slip plane. This defines a dilation angle ψ which as before depends on θ and the internal variables, i.e.

$$\psi = \psi(\theta - \theta_p, A, B) \tag{1.7}$$

- *Hardening rule*: A hardening rule here means that $\dot{A}$, $\dot{B}$ and $\dot{\theta}_p$, are functions of their current value, of the incremental plastic strain $\dot{\varepsilon}_p$, but also of the rotation $\dot{\omega}$, i.e. the antisymmetric part of velocity gradient, because of the induced anisotropic texture of the medium. This hardening law should account for the advection of contacts in the plastic flow, as well as the creation and opening of contacts. A last constraint to be taken into account comes from the quasistatic nature of the loading we are interested in. This implies that the rheology should be rate independent, i.e. time as such should not play a role in those equations. Thus $\dot{A}$ (as well as all other such rates) should depend on $\dot{\varepsilon}_p$ through a positively homogeneous function of degree 1.

As in the previous case, there are now different routes to follow according to different strategies. Again, one possible route is a purely phenomenological approach with the perspective of identifying all the required information from tests. There still some basic principles should be used to constraint the hardening rule. However, due to the crowding of internal variables this task is quite challenging. A second route is to circumvent first the form of the possible dependencies by means of expansions around special values of the internal variables, such as the isotropic case $B = 0$ or, alternatively, around values which would correspond to the "critical states". We use the latter in plural form because the critical state may well depend, for instance, on the relative rotation rate with respect to the deviatoric strain rate, i.e. the critical state under pure shear may differ from the critical state under simple shear. To be more precise, we may assume that, if the deviatoric part of the fabric tensor is reasonably small, ϕ may be expanded as

$$\phi(\theta - \theta_p, A, B) = a_0 + a_1 A + a_2 B \cos(2(\theta - \theta_p) + a_3) \qquad (1.8)$$

so that the determination of the full function of three parameters is reduced to four scalar parameters a_i for $i = 0, \ldots, 3$.

Another possible strategy is to resort to the microscopic world in order to identify the above-listed unknown functions. We now discuss some elements along this direction.

6. MICRO-MACRO TRANSITION

At the macroscopic level, we deal with stress and strain as continuous fields. However, at the particle level, we have a discrete set of forces transmitted by the interparticle contacts and velocities of particles. The connection between these two levels of description has been the subject of intensive work. A clear review of different proposed relations has been presented by Bardet.(Bardet 98) Let us first recall very briefly a few essential results.

6.1. FROM DISCRETE TO CONTINUOUS

The first key point is to define the average stress over any domain which encompasses an arbitrary number of particles. This can be done consistently down to the scale of one single particle. Let us label each contact around one particle by an index i, and denote the unit normal vector by $\vec{n}_i$ and the force transmitted at the contact by $\vec{f}_i$. The stress which characterizes the particle is(Cundall and Strack 79)

$$\sigma = \frac{R}{S} \sum_i \vec{n}_i \otimes \vec{f}_i \qquad (1.9)$$

where S is an area associated with the particle from a Voronoï construction from particle centers. This expression exploits the fact that the particles are at rest and no torque is transmitted at the contacts. We have assumed that the particles are circular with a radius R. We see that the transition from forces to stress requires the definition of a representative environment of a particle where each neighbor is specified, i.e. a *"node"* of the contact network.

The analogous treatment of the displacement field is somewhat more subtle. The main point is that the object in which we are interested at the macroscopic scale is the displacement field particle centers. Whether a particle rotates or not will affect the velocity of a material element in the grain, but not the overall displacement. Thus, the strategy usually adopted in this respect is to construct a continuous field from an interpolation of the velocity field which matches exactly the particle velocity at the particle center.

The standard technique is again to resort to a tesselation of space with polygons whose vortices are the particle centers, and whose edges connect the contacting particles. Once the problem is reduced to the estimation of the average strain inside an elementary polygon whose boundary velocity is prescribed, the solution is standard.(Kruyt, 1996) As in the case of force/stress relation, here we need a representative elementary structure which is a polygon formed by contiguous particles, what will be refered to as a *"cell"* in the sequel.

6.2. ENVIRONMENTS: NODES AND CELLS

A major difficulty is that, as discussed above, the macroscopic description of the geometrical state of a granular medium is based on the fabric represented by $p(\theta)$ (including a more or less severe truncation). However, in order to construct the two elementary tensors, we have to deal with nodes or cells and this requires a richer information. For instance, a node is characterized by its coordination number z and the orientations $\theta_1, \cdots, \theta_z$ of its contact normals. Thus, we need the probability distributions $p_z(\theta_1, ..., \theta_z)$. The corresponding issue is trivially written in similar terms for the cells.

In order to construct these representative environments, we have to propose an "educated guess" for these multicontact distributions. An easy solution is to assume the most "disordered" situation, i.e.

$$p_z(\theta_1, ..., \theta_z) = \prod_{i=1}^{z} p(\theta_i) \qquad (1.10)$$

In our case, this solution is obviously wrong since a contact in the direction θ_i with a given particle impedes other contacts to be established in a direction θ_j such that $|\theta_i - \theta_j| < \pi/3$, hence

$$p_z(\theta_1, ..., \theta_z) = 0 \qquad \text{if} \qquad |\theta_i - \theta_j| < \pi/3 \qquad (1.11)$$

Here, the strategy that we propose is still to resort to a similar "most disordered" situation, provided these steric constraints are taken into account. Operationnally, this translates into the maximization of an entropy functionnal $S[p_z]$ under constraints. The entropy, S, is classically defined as

$$S[p_z] = \int ... \int p_z(\{\theta_i\}) \log(p_z(\{\theta_i\}))d\{\theta_i\} \tag{1.12}$$

and S is maximized over the set of functions that 1) fulfill the steric hindrance conditions, Eq. 1.11, and 2) whose partial summation over all but one angle gives back the known $p(\theta)$. The second constraint is imposed through Lagrange multipliers. Let us introduce $H(\theta)$, a 2π-periodic function such that $H(\theta) = 0$ for $|\theta| < \pi/3$, and else $H = 1$. It can be shown that p_z takes the following form:

$$p_z(\{\theta_i\}) = \left(\prod_{i \neq j} H(\theta_i - \theta_j)\right) \prod_k g(\theta_k) \tag{1.13}$$

The unknown function g is then determined from an implicit equation resulting from the resummation condition (2), and whose solution can be obtained from a simple iterative scheme. Note that without steric hindrance the simple solution $p_z = \prod p(\theta_i)$ is recovered.

We note that not all values of A and B in the truncated form of p admit a solution. The condition is that

$$\int_{\theta-\pi/6}^{\theta+\pi/6} p(\theta) d\theta \leq 1 \tag{1.14}$$

which simply states that no more than one contact can be found in any sector of opening angle $\pi/3$. Using the simple truncated form Eq. 1.5, we obtain

$$\frac{\pi}{6} A + \frac{\sqrt{3}}{4} B \leq 1 \tag{1.15}$$

which together with the condition $B \leq A$ (no negative probability), give the domain of physically accessible values of the fabric.

7. YIELD STRESS AND PLASTIC STRAIN DIRECTIONS

The previous section proposed an operational way of generating representative nodes (and using a similar construction, cells) with acceptable statistics, i.e. consistent with the known information concerning fabric. Thus, we have now a key which will allow us to compute the yield stress. From the previous discussion, we have to compute the maximum allowed deviatoric stress for all orientations of the principal axes of stress with respect to that of the fabric. We propose a Monte-Carlo procedure, which consists in generating a large collection of node configurations with the computed statistics. Then, for fixed orientation and trace of the stress tensor we compute the maximum value of its deviatoric component that can be obtained from the contact forces with the following constraints: 1) The forces are balanced; 2) They yield the imposed stress when using Eq. 1.9; 3) Each force fulfills Signorini (no traction) and Coulomb conditions. A simple average over the configurations gives an estimate of the searched yield surface.

A similar procedure can be designed for the strain. Cells are generated, and for each configuration the minimal dilation angle for a principal strain orientation is computed. This is done by computing an admissible set of particle center velocities that is consistent with the average strain, and with the steric exclusion conditions. An average over configurations gives the searched dilation angle.

8. HARDENING RULE

We still have to address the final point which is the hardenning rule, i.e. the evolution of the fabric, in terms of $p(\theta)$ or fabric tensors as a function of strain. The two effects to take into account are 1) the advection of contacts by the plastic flow, 2) the induction (gain or loss) of contacts.(Roux and Radjaï) We can write the time evolution of $p(\theta)$ as a balance equation:

$$\frac{\partial p(\theta)}{\partial t} + \mathrm{div}(J(\theta)) = I(\theta) \tag{1.16}$$

where the divergence operator is simply $\partial/\partial\theta$, J is the contact "current" and I the induction term. The mean velocity field with respect to a particle center in polar coordinates is written $(u(r,\theta), v(r,\theta))$. For contacting particles, where $r = 2R$, the mean velocities are written $(U(\theta), V(\theta))$. The expression of the contact current is thus simply written as

$$J(\theta) = p(\theta)\frac{V(\theta)}{2R} \tag{1.17}$$

The Signorini condition requires that $U \geq 0$. Galilean invariance dictates that the effect of a rotation rate $\dot{\omega}$ should come into play only in the tangential component, as $V(\theta) = \delta V(\theta) + 2R\dot{\omega}$, and $v(r,\theta) = \delta v(r,\theta) + r\dot{\omega}$. We note that current can also be split in two contributions, $J = J_r + J_s$, one part from the rotation, and one from the pure deformation term. Extracting the rotation term from the current, we can rewrite the l.h.s. expression of the balance equation as

$$\frac{\partial p(\theta)}{\partial t} + \dot{\omega} = \frac{\partial p(\theta)}{\partial \theta} + \mathrm{div}(J_s(\theta)) = \frac{\mathrm{d}p(\theta)}{\mathrm{d}t} + \mathrm{div}(J_s(\theta)) \tag{1.18}$$

where the total time derivative includes the rotation term. With this in mind, we can now safely ignore the rotation effect.

The velocity field for non contacting particles can be deduced simply from a mean-field assumption since no direct steric constraints are at play. Thus u and δv are directly related to $\dot{\varepsilon}_p$.

The induction term I consists of a creation I_+ of contacts, and an opening term I_-, with $I = I_+ - I_-$. The creation of contacts involves the non-contacting normal velocity when the latter is negative, and the probability that a given particle lies in the region sufficiently close to reach the reference particle. This involves the packing fraction, through the areal density of centers $c/(\pi R^2)$. The creation term is written

$$I_+(\theta) = -\frac{2c}{\pi R}[u(2R,\theta)]_- \tag{1.19}$$

where $[...]_-$ denotes the negative part of the velocity. The contact opening term is

$$I_-(\theta) = p(\theta)U(\theta) \tag{1.20}$$

Up to now, we have specified all terms except the contacting velocities $(U, \delta V)$. They are obviously related to $(u, \delta v)$ through a geometric function which is positively homogeneous of degree one, because of the rate independence. This could be computed together with the direction of the plastic strain rate, because at this stage we generate a representative set of local configurations where we have access to the relative particle velocities.

Alternatively, we may consider two arbitrarily remote particles along a given direction, which can be reached through a path of contacting particles. When the particles are sufficiently far from each other, their relative velocity is equal to the macroscopically determined one. However, it is also equal to the sum of the relative contacting particles along the path. This contraints $(U, \delta V)$ to be equal to $(u, \delta v)$ when averaged over an interval of angles which allows for the existence of a path. This allows the deficit in negative U values to be compensated by a larger value of V at a different angle. The difference between δV and δv, i.e. the deviation from mean field, may be interpreted as some kind of diffusive contribution to the current due to the steric hindrance transmitted through particles outside our "shell" description.

This section has been written directly in terms of the entire distribution $p(\theta)$ and not its truncated Fourier expansion. In order to express the hardening equations in terms of A, B and θ_p, it suffices to consider weak formulations of the latter by multiplying Eq. 1.16, by 1, $\cos(2\theta)$ and $\sin(2\theta)$, and integrate over all angles θ. Then, a term-by-term identification gives a direct transcription corresponding to the reduced description of the fabric.

Let us finally mention an additional important point. We pointed out that the fabric $p(\theta)$ is restricted by geometrical constraints (see e.g. Eq.1.15) at the particle level. However, these constraints do not appear in the present hardening rules so that they might be violated if one proceeds along the proposed route. A consistent way to circumvent this difficulty is to require that the fabric always lies in the admissible domain. This will in turn generate an admissible set of plastic strain increments. The latter may thus be used to define the plastic flow rule. This would constitute an alternative way to end up with a consistent rigid-plastic formulation, and it would by-pass the stage which consists in generating representative "cells". In other words, it would exploit different ways of relating the relative velocities of contacting particles to the mean strain field, through paths rather than elementary cells.

9. EFFECTS OF FLUCTUATIONS

In this section, we briefly discuss potential limitations to the scheme proposed above, as inferred from numerical observations. Most of these difficulties result from fluctuations either in time or in space that have received in the past a much more limited attention than average behaviors. Indeed, the above scheme is based at each time step on rest configurations. However, when running numerical simulations we observe that from time to time the system reaches unstable points where a dynamical rearrangement of particles has to take place. In this dynamical phase, inelastic collisions dissipate the potential energy drop, (i.e. the work of the loading forces), due to the reorganization of the assembly.

The cumulative effect of these restructuring events may have paradoxical effects. In particular they can produce an effective mean dissipation which is "Coulomb"-like even for ideal frictionless particles. Indeed, we can imagine that averaging the instantaneous dilation rate during a long steady shear strain (once the steady state has been reached) over all rest configurations, we may obtain a positive value which is exactly counterbalanced by the compression taking place during these unstable events. The dynamics of restructuring being much faster than any external loading (quasistatic condition), the (time averaged) dissipation appears to be rate independent. Moreover, the dissipation at each event is equal to the external work accumulated prior to the event. Therefore, the dissipation is simply proportional to the external loading. Both of these features characterize a solid "Coulomb" friction.

It is amusing to note that a similar explanation based on elastic asperity interactions is at the heart of microscopic physical modelling of friction.(Caroli 96; Tanguy 98) Trying to quantify the above effect in a more realistic case, we quantified the relative part of dissipation due to inelastic collisions, as compared to the total dissipation, i.e. including contact friction (for an interparticle coefficient of friction $\mu = 0.5$). The resulting ratio was close to 30% due to retructuring events. This effect being absent from the proposed description, the effective macroscopic friction angle is at best off by this amount if a global dissipation balance is performed. In contrast, an analysis of experimental results based on instantaneous values of the stress may be in much closer agreement.

Fluctuations may be important for basically two reasons. The first one is their amplitude relative to the mean, and the second is the existence of long-range correlations which may be present either in time or space and which may affect very significantly the validity of the proposed approach. In order to investigate the latter, we studied the trajectory of particles in a simple shear test. The analysis of the fluctuating part of the displacement and of the distance between two neighboring particles reveals the existence of long-range temporal fluctuation which give rise to an anomalous diffusion regime. Similarly, the instantaneous strain field displays also large scale inhomogeneity which were unexpected. These observations which are not taken into account in the approach we presented may potentially invalidate a quantitative agreement between theory and numerical or experimental tests.

10. CONCLUSIONS

We presented a systematic approach to progressively enrich a description of the mechanical behavior of a granular medium both from macroscopic and microscopic (particle level) points of view. This constitutes a template and a number of these elementary steps have still to be validated in particular from detailed numerical simulations. We proposed an original scheme based on entropy maximization that allowes to generate representative environments around particles, an essential step for determination of the yield stress or the plastic strain rate direction. We also proposed a simple "shell model" description of the velocity field around a particle which allowes to obtain an evolution equation for the fabric.

Finally, we pointed out some potential difficulties of this kind of approach where salient aspects of fluctuations, not included in the theoretical approach, may affect the accuracy of the description. Nevertheless, the global framework, i.e. the form of the equations, should be unaffected. A macroscopic description of these fluctuations might thus appear a necessary step for a complete description, and that is, in our view, a major challenge for the future.

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